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On the Carathéodory Form in Higher-Order Variational Field Theory

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Abstract: The Carathéodory form of the calculus of variations belongs to the class of Lepage equivalents of first-order Lagrangians in field theory. Here, this equivalent is generalized for second- and higher-order Lagrangians by means of intrinsic geometric operations applied to the well-known Poincaré–Cartan form and principal component of Lepage forms, respectively. For second-order theory, our definition coincides with the previous result obtained by Crampin and Saunders in a different way. The Carathéodory equivalent of the Hilbert Lagrangian in general relativity is discussed.

Keywords: Carathéodory form; Poincaré–Cartan form; Lepage equivalent; fibered manifold; variational field theory

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1. Introduction

In this note, we describe a generalization of the Carathéodory form [1] (cf. [2–5]) of the calculus of variations in *second-order* and, for specific Lagrangians, in *higher-order* field theory.

The theory of first-order multiple-integral variational functionals for unknown sections of fibered spaces over n-dimensional base manifolds can be described in a coordinate-independent way by means of the Poincaré–Cartan form [6] or its extensions, namely the Carathéodory form and the fundamental Lepage equivalent (also known as the Krupka–Betounes form; see [7,8]). For a comprehensive exposition of the theory as well as original references, see the monographs in [9,10].

The above-mentioned differential forms represent examples of *Lepage forms*—a far-going generalization of a 1-form, introduced by Cartan in the 1920s within the framework of the calculus of variations of simple integrals and classical mechanics. The generalization, given by Krupka [11,12], is motivated by the work of Lepage in the 1940s. We can briefly say that these objects define the same variational functional as it is prescribed by a given Lagrangian and, moreover, variational properties (as variations, extremals, or Noether's type invariance) of the corresponding functional are globally characterized in terms of geometric operations (such as the exterior derivative and the Lie derivative) acting on integrands—the Lepage equivalents of a Lagrangian. In particular, the extremals are determined by means of the exterior derivative of the Lepage form. The meaning of Lepage forms for the calculus of variations and their basic properties, including examples of Lepage forms in geometric mechanics and field theory, have recently been summarized and further studied [13–19].

Our approach towards generalizing the Carathéodory form is based on a geometric relationship between the Poincaré–Cartan and Carathéodory forms, and analysis of the corresponding global properties. In [3], Crampin and Saunders obtained the Carathéodory form for second-order Lagrangians as a certain projection onto a sphere bundle. Here, we confirm this result by means of a different, straightforward method which furthermore allows higher-order generalization. It is a standard fact in the global variational field theory that the local expressions,

$$\Theta_\lambda = \mathscr{L}\omega_0 + \sum_{k=0}^{r-1}\left(\sum_{l=0}^{r-1-k}(-1)^l d_{p_1}\dots d_{p_l}\frac{\partial\mathscr{L}}{\partial y^\sigma_{j_1\dots j_k p_1\dots p_l i}}\right)\omega^\sigma_{j_1\dots j_k}\wedge\omega_i, \tag{1}$$

which generalize the Poincaré–Cartan form of the calculus of variations, define (1), in general, *globally* for Lagrangians $\lambda = \mathscr{L}\omega_0$ of order $r = 1$ and $r = 2$ *only*; see the works of Krupka [12] (Θ_λ is known as the principal component of a Lepage equivalent of Lagrangian λ) and Horák and Kolář [20] (for higher-order Poincaré–Cartan morphisms). We show that, if Θ_λ is a globally defined differential form for a *class* of Lagrangians of order $r \geq 3$, then a higher-order Carathéodory equivalent for Lagrangians belonging to this class naturally arises by means of geometric operations acting on Θ_λ. To this purpose, for order $r = 3$, we analyze conditions, which describe the obstructions for globally defined principal components of Lepage equivalents (1) (or, higher–order Poincaré–Cartan forms).

A concrete application of our result in second-order field theory includes the Carathéodory equivalent of the Hilbert Lagrangian in general relativity.

Basic underlying structures, well adapted to this paper, can be found in the work by Volná and Urban [18]. If (U, φ), $\varphi = (x^i)$, is a chart on a smooth manifold X, we set

$$\omega_0 = dx^1\wedge\dots\wedge dx^n, \qquad \omega_j = i_{\partial/\partial x^j}\omega_0 = \frac{1}{(n-1)!}\varepsilon_{ji_2\dots i_n}dx^{i_2}\wedge\dots\wedge dx^{i_n},$$

where $\varepsilon_{i_1 i_2\dots i_n}$ is the Levi–Civita permutation symbol. If $\pi : Y \to X$ is a fibered manifold and W an open subset of Y, then there exists a unique morphism $h : \Omega^r W \to \Omega^{r+1}W$ of exterior algebras of differential forms such that for any fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, where $V \subset W$, and any differentiable function $f : W^r \to \mathbb{R}$, where $W^r = (\pi^{r,0})^{-1}(W)$ and $\pi^{r,s} : J^rY \to J^sY$ the jet bundle projection,

$$hf = f\circ\pi^{r+1,r}, \qquad hdf = (d_i f)dx^i,$$

where

$$d_i = \frac{\partial}{\partial x^i} + \sum_{j_1\leq\dots\leq j_k}\frac{\partial}{\partial y^\sigma_{j_1\dots j_k}}y^\sigma_{j_1\dots j_k i} \tag{2}$$

is the ith formal derivative operator associated with (V, ψ). A differential form q-form $\rho \in \Omega^r_q W$ satisfying $h\rho = 0$ is called *contact*, and ρ is generated by contact 1-forms

$$\omega^\sigma_{j_1\dots j_k} = dy^\sigma_{j_1\dots j_k} - y^\sigma_{j_1\dots j_k s}dx^s, \qquad 0 \leq k \leq r-1.$$

Throughout, we use the standard geometric concepts: the exterior derivative d, the contraction $i_\Xi\rho$ and the Lie derivative $\partial_\Xi\rho$ of a differential form ρ with respect to a vector field Ξ, and the pull-back operation $*$ acting on differential forms.

2. Lepage Equivalents in First- and Second-Order Field Theory

By a *Lagrangian* λ for a fibered manifold $\pi : Y \to X$ of order r we mean an element of the submodule $\Omega^r_{n,X}W$ of π^r-horizontal n-forms in the module of n-forms $\Omega^r_n W$, defined on an open subset W^r of the rth jet prolongation J^rY. In a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, where $V \subset W$, the Lagrangian $\lambda \in \Omega^r_{n,X}W$ has an expression

$$\lambda = \mathscr{L}\omega_0, \tag{3}$$

where $\omega_0 = dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$ is the (local) volume element, and $\mathscr{L} : V^r \to \mathbb{R}$ is the *Lagrange function* associated to λ and (V, ψ).

An n-form $\rho \in \Omega^s_n W$ is called a *Lepage equivalent* of $\lambda \in \Omega^r_{n,X}W$, if the following two conditions are satisfied:

(i) $(\pi^{q,s+1})^* h\rho = (\pi^{q,r})^* \lambda$ (i.e., ρ is *equivalent* with λ); and
(ii) $hi_\xi d\rho = 0$ for arbitrary $\pi^{s,0}$-vertical vector field ξ on W^s (i.e., ρ is a *Lepage form*).

The following theorem describes the structure of the Lepage equivalent of a Lagrangian (see [9,12]).

Theorem 1. *Let $\lambda \in \Omega^r_{n,X} W$ be a Lagrangian of order r for Y, locally expressed by (3) with respect to a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$. An n-form $\rho \in \Omega^s_n W$ is a Lepage equivalent of λ if and only if*

$$(\pi^{s+1,s})^* \rho = \Theta_\lambda + d\mu + \eta, \tag{4}$$

where the n-form Θ_λ is defined on V^{2r-1} by (1), μ is a contact $(n-1)$-form, and an n-form η has the order of contactness ≥ 2.

Θ_λ is called the *principal component* of the Lepage form ρ with respect to fibered chart (V, ψ). In general, decomposition (4) is *not* uniquely determined with respect to contact forms μ, η, and the principal component Θ_λ need *not* define a global form on W^{2r-1}. Nevertheless, the Lepage equivalent ρ satisfying (4) is globally defined on W^s; moreover, $E_\lambda = p_1 d\rho$ is a globally defined $(n+1)$-form on W^{2r}, called the *Euler–Lagrange form* associated to λ.

We recall the known examples of Lepage equivalents of first- and second-order Lagrangians, determined by means of additional requirements.

Lemma 1. (Principal Lepage form)

(a) *For every Lagrangian λ of order $r = 1$, there exists a unique Lepage equivalent Θ_λ of λ on W^1, which is $\pi^{1,0}$-horizontal and has the order of contactness ≤ 1. In a fibered chart (V, ψ), Θ_λ has an expression*

$$\Theta_\lambda = \mathscr{L}\omega_0 + \frac{\partial \mathscr{L}}{\partial y^\sigma_j} \omega^\sigma \wedge \omega_j. \tag{5}$$

(b) *For every Lagrangian λ of order $r = 2$, there exists a unique Lepage equivalent Θ_λ of λ on W^3, which is $\pi^{3,1}$-horizontal and has the order of contactness ≤ 1. In a fibered chart (V, ψ), Θ_λ has an expression*

$$\Theta_\lambda = \mathscr{L}\omega_0 + \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_j} - d_p \frac{\partial \mathscr{L}}{\partial y^\sigma_{pj}} \right) \omega^\sigma \wedge \omega_j + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \omega^\sigma_i \wedge \omega_j. \tag{6}$$

For $r = 1$ and $r = 2$, the principal component Θ_λ (1) is a *globally defined* Lepage equivalent of λ. We point out that for $r \geq 3$ this is *not* true (see [12,20]). (5) is the well-known *Poincaré–Cartan form* (cf. García [6]), and it is generalized for second-order Lagrangians by the globally defined *principal Lepage equivalent* (6) on $W^3 \subset J^3 Y$.

Lemma 2 (Fundamental Lepage form). *Let $\lambda \in \Omega^1_{n,X} W$ be a Lagrangian of order 1 for Y, locally expressed by (3). There exists a unique Lepage equivalent $Z_\lambda \in \Omega^1_n W$ of λ, which satisfies $Z_{h\rho} = (\pi^{1,0})^* \rho$ for any n-form $\rho \in \Omega^0_n W$ on W such that $h\rho = \lambda$. With respect to a fibered chart (V, ψ), Z_λ has an expression*

$$Z_\lambda = \mathscr{L}\omega_0 + \sum_{k=1}^{n} \frac{1}{(n-k)!} \frac{1}{(k!)^2} \frac{\partial^k \mathscr{L}}{\partial y^{\sigma_1}_{j_1} \dots \partial y^{\sigma_k}_{j_k}} \varepsilon_{j_1 \dots j_k i_{k+1} \dots i_n} \cdot \omega^{\sigma_1} \wedge \dots \wedge \omega^{\sigma_k} \wedge dx^{i_{k+1}} \wedge \dots \wedge dx^{i_n}. \tag{7}$$

Z_λ (7) is known as the *fundamental Lepage form* [7,8]), and it is characterized by the equivalence: Z_λ is closed if and only if λ is trivial (i.e., the Euler–Lagrange expressions associated with λ vanish identically). Recently, the form (7) was generalized in [21], and studied for variational problems for submanifolds in [5], as well as applied for studying symmetries and conservation laws in [22].

Lemma 3 (**Carathéodory form**). *Let $\lambda \in \Omega^1_{n,X}W$ be a non-vanishing Lagrangian of order 1 for Y (3). Then, a differential n-form $\Lambda_\lambda \in \Omega^1_n W$, locally expressed as*

$$\begin{aligned}\Lambda_\lambda &= \frac{1}{\mathscr{L}^{n-1}} \bigwedge_{j=1}^{n} \left(\mathscr{L} dx^j + \frac{\partial \mathscr{L}}{\partial y^\sigma_j} \omega^\sigma \right) \\ &= \frac{1}{\mathscr{L}^{n-1}} \left(\mathscr{L} dx^1 + \frac{\partial \mathscr{L}}{\partial y^{\sigma_1}_1} \omega^{\sigma_1} \right) \wedge \ldots \wedge \left(\mathscr{L} dx^n + \frac{\partial \mathscr{L}}{\partial y^{\sigma_n}_n} \omega^{\sigma_n} \right),\end{aligned} \tag{8}$$

is a Lepage equivalent of λ.

Λ_λ (7) is the well-known *Carathéodory form* (cf. [1]), associated to Lagrangian $\lambda \in \Omega^1_{n,X}W$, which is nowhere zero. Λ_λ is uniquely characterized by the following properties: Λ_λ is: (i) a Lepage equivalent of λ; (ii) decomposable; and (iii) $\pi^{1,0}$-horizontal (i.e., semi-basic with respect to projection $\pi^{1,0}$).

3. The Carathéodory Form: Second-Order Generalization

Let $\lambda \in \Omega^1_{n,X}W$ be a *non-vanishing, first-order* Lagrangian on $W^1 \subset J^1Y$. In the next lemma, we describe a new observation, showing that the Carathéodory form Λ_λ (8) arises from the Poincaré–Cartan form Θ_λ (5) by means of contraction operations on differential forms with respect to the formal derivative vector fields d_i (2).

Lemma 4. *The Carathéodory form Λ_λ (8) and the Poincaré–Cartan form Θ_λ (5) satisfy*

$$\Lambda_\lambda = \frac{1}{\mathscr{L}^{n-1}} \bigwedge_{j=1}^{n} (-1)^{n-j} i_{d_n} \ldots i_{d_{j+1}} i_{d_{j-1}} \ldots i_{d_1} \Theta_\lambda.$$

Proof. From the decomposable structure of Λ_λ, we see that what is needed to show is the formula

$$i_{d_n} \ldots i_{d_{j+1}} i_{d_{j-1}} \ldots i_{d_1} \Theta_\lambda = (-1)^{n-j} \left(\mathscr{L} dx^j + \frac{\partial \mathscr{L}}{\partial y^\sigma_j} \omega^\sigma \right)$$

for every j, $1 \leq j \leq n$. Since $dx^k \wedge \omega_j = \delta^k_j \omega_0$, the Poincaré–Cartan form is expressible as

$$\begin{aligned}\Theta_\lambda &= \mathscr{L}\omega_0 + \frac{\partial \mathscr{L}}{\partial y^\sigma_j} \omega^\sigma \wedge \omega_j = \mathscr{L}\omega_0 + \frac{\partial \mathscr{L}}{\partial y^\sigma_j} dy^\sigma \wedge \omega_j - \frac{\partial \mathscr{L}}{\partial y^\sigma_j} y^\sigma_k dx^k \wedge \omega_j \\ &= \left(\mathscr{L} - \frac{\partial \mathscr{L}}{\partial y^\sigma_1} y^\sigma_1 - \frac{\partial \mathscr{L}}{\partial y^\sigma_2} y^\sigma_2 - \ldots - \frac{\partial \mathscr{L}}{\partial y^\sigma_n} y^\sigma_n \right) \omega_0 + \frac{\partial \mathscr{L}}{\partial y^\sigma_j} dy^\sigma \wedge \omega_j.\end{aligned}$$

Applying the contraction operations to Θ_λ, we obtain by means of a straightforward computation for every j,

$$\begin{aligned}i_{d_{j-1}} \ldots i_{d_1} \Theta_\lambda = &\left(\mathscr{L} - \frac{\partial \mathscr{L}}{\partial y^\sigma_j} y^\sigma_j - \ldots - \frac{\partial \mathscr{L}}{\partial y^\sigma_n} y^\sigma_n \right) dx^j \wedge \ldots \wedge dx^n \\ &+ (-1)^{j-1} \sum_{k=j}^{n} \frac{\partial \mathscr{L}}{\partial y^\sigma_k} dy^\sigma \wedge i_{d_{j-1}} \ldots i_{d_1} \omega_k \\ &+ \sum_{l=1}^{j-1} (-1)^{l-1} y^\sigma_l i_{d_{j-1}} \ldots i_{d_{l+1}} i_{d_{l-1}} \ldots i_{d_1} \left(\sum_{k=j}^{n} \frac{\partial \mathscr{L}}{\partial y^\sigma_k} \omega_k \right),\end{aligned}$$

and

$$
\begin{aligned}
& i_{d_{j+1}} i_{d_{j-1}} \dots i_{d_1} \Theta_\lambda \\
& \quad = -\left(\mathscr{L} - \frac{\partial \mathscr{L}}{\partial y^\sigma_j} y^\sigma_j - \sum_{k=j+2}^{n} \frac{\partial \mathscr{L}}{\partial y^\sigma_k} y^\sigma_k \right) dx^j \wedge dx^{j+2} \wedge \dots \wedge dx^n \\
& \quad + (-1)^j \sum_{k=j}^{n} \frac{\partial \mathscr{L}}{\partial y^\sigma_k} dy^\sigma \wedge i_{d_{j+1}} i_{d_{j-1}} \dots i_{d_1} \omega_k \\
& \quad + \sum_{l=1}^{j-1} (-1)^{l-1} y^\sigma_l i_{d_{j+1}} i_{d_{j-1}} \dots i_{d_{l+1}} i_{d_{l-1}} \dots i_{d_1} \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_j} \omega_j + \sum_{k=j+2}^{n} \frac{\partial \mathscr{L}}{\partial y^\sigma_k} \omega_k \right) \\
& \quad + (-1)^{j-1} y^\sigma_{j+1} i_{d_{j-1}} \dots i_{d_1} \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_j} \omega_j + \sum_{k=j+2}^{n} \frac{\partial \mathscr{L}}{\partial y^\sigma_k} \omega_k \right).
\end{aligned}
$$

Following the inductive structure of the preceding expressions, we get after the next $n-j-1$ steps,

$$
\begin{aligned}
& i_{d_n} \dots i_{d_{j+1}} i_{d_{j-1}} \dots i_{d_1} \Theta_\lambda \\
& \quad = (-1)^{n-j} \left(\mathscr{L} - \frac{\partial \mathscr{L}}{\partial y^\sigma_j} y^\sigma_j \right) dx^j + (-1)^{n-j} \frac{\partial \mathscr{L}}{\partial y^\sigma_j} dy^\sigma - (-1)^{n-j} \frac{\partial \mathscr{L}}{\partial y^\sigma_j} \sum_{k \neq j} y^\sigma_k dx^k \\
& \quad = (-1)^{n-j} \left(\mathscr{L} dx^j + \frac{\partial \mathscr{L}}{\partial y^\sigma_j} \omega^\sigma \right),
\end{aligned}
$$

as required. □

An intrinsic nature of Lemma 4 indicates a possible extension of the Carathéodory form (8) for higher-order variational problems. We put

$$
\Lambda_\lambda = \frac{1}{\mathscr{L}^{n-1}} \bigwedge_{j=1}^{n} (-1)^{n-j} i_{d_n} \dots i_{d_{j+1}} i_{d_{j-1}} \dots i_{d_1} \Theta_\lambda, \tag{9}
$$

where Θ_λ in (9) denotes the principal Lepage equivalent (6) of a *second-order* Lagrangian λ, and verify that formula (9) defines a global form.

Theorem 2. *Let $\lambda \in \Omega^2_{n,X} W$ be a non-vanishing second-order Lagrangian on $W^2 \subset J^2 Y$. Then, Λ_λ satisfies:*

(a) *Formula (9) defines an n-form on $W^3 \subset J^3 Y$.*

(b) *If $\lambda \in \Omega^2_{n,X} W$ has an expression (3) with respect to a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, such that $V \subset W$, then Λ_λ (9) is expressed by*

$$
\Lambda_\lambda = \frac{1}{\mathscr{L}^{n-1}} \bigwedge_{j=1}^{n} \left(\mathscr{L} dx^j + \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_j} - d_i \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \right) \omega^\sigma + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \omega^\sigma_i \right). \tag{10}
$$

(c) *$\Lambda_\lambda \in \Omega^3_n W$ (9) associated to a second-order Lagrangian $\lambda \in \Omega^2_{n,X} W$ is a Lepage equivalent of λ, which is decomposable and $\pi^{3,1}$-horizontal.*

Proof.

1. Suppose (V, ψ), $\psi = (x^i, y^\sigma)$, and $(\bar{V}, \bar{\psi})$, $\bar{\psi} = (\bar{x}^i, \bar{y}^\sigma)$, are two overlapping fibered charts on W. For $\lambda \in \Omega^2_{n,X} W$, the corresponding chart expressions $\lambda = \mathscr{L} \omega_0$ and $\lambda = \bar{\mathscr{L}} \bar{\omega}_0$ satisfy

$$
\mathscr{L} = \left(\bar{\mathscr{L}} \circ \bar{\psi}^{-1} \circ \psi \right) \det \frac{\partial \bar{x}^i}{\partial x^j}. \tag{11}
$$

Since the push-forward vector field $\bar{d}_k$,

$$\bar{d}_k = \frac{\partial}{\partial \bar{x}^k} + \bar{y}^\sigma_k \frac{\partial}{\partial \bar{y}^\sigma} + \bar{y}^\sigma_{kl} \frac{\partial}{\partial \bar{y}^\sigma_l},$$

of vector field $(\partial x^i / \partial \bar{x}^k) d_i$ with respect to the chart transformation $\bar{\psi}^{-1} \circ \psi$ satisfies

$$(\bar{\psi}^{-1} \circ \psi)^* \left(i_{\bar{d}_k} \Theta_\lambda \right) = i_{\frac{\partial x^i}{\partial \bar{x}^k} d_i} \left((\bar{\psi}^{-1} \circ \psi)^* \Theta_\lambda \right) = \frac{\partial x^i}{\partial \bar{x}^k} i_{d_i} \left((\bar{\psi}^{-1} \circ \psi)^* \Theta_\lambda \right),$$

and Θ_λ (6) is globally defined, we get

$$\begin{aligned}
&(\bar{\psi}^{-1} \circ \psi)^* \frac{1}{\bar{\mathscr{L}}^{n-1}} \bigwedge_{j=1}^{n} (-1)^{n-j} i_{\bar{d}_n} \dots i_{\bar{d}_{j+1}} i_{\bar{d}_{j-1}} \dots i_{\bar{d}_1} \Theta_\lambda \\
&= \frac{1}{(\bar{\mathscr{L}} \circ \bar{\psi}^{-1} \circ \psi)^{n-1}} \bigwedge_{j=1}^{n} (-1)^{n-j} \frac{\partial x^{i_1}}{\partial \bar{x}^1} \dots \frac{\partial x^{i_{j-1}}}{\partial \bar{x}^{j-1}} \frac{\partial x^{i_{j+1}}}{\partial \bar{x}^{j+1}} \dots \frac{\partial x^{i_n}}{\partial \bar{x}^n} \\
&\quad \cdot i_{d_{i_n}} \dots i_{d_{i_{j+1}}} i_{d_{i_{j-1}}} \dots i_{d_{i_1}} \left((\bar{\psi}^{-1} \circ \psi)^* \Theta_\lambda \right) \\
&= \frac{1}{(\bar{\mathscr{L}} \circ \bar{\psi}^{-1} \circ \psi)^{n-1}} \left(\frac{\partial x^{i_1}}{\partial \bar{x}^1} \dots \frac{\partial x^{i_n}}{\partial \bar{x}^n} \varepsilon_{i_1 \dots i_n} \right)^n \\
&\quad \cdot \bigwedge_{j=1}^{n} (-1)^{n-j} i_{d_n} \dots i_{d_{j+1}} i_{d_{j-1}} \dots i_{d_1} (\bar{\psi}^{-1} \circ \psi)^* \Theta_\lambda \\
&= \frac{1}{\left((\bar{\mathscr{L}} \circ \bar{\psi}^{-1} \circ \psi) \det \frac{\partial \bar{x}^i}{\partial x^j} \right)^{n-1}} \bigwedge_{j=1}^{n} (-1)^{n-j} i_{d_n} \dots i_{d_{j+1}} i_{d_{j-1}} \dots i_{d_1} (\bar{\psi}^{-1} \circ \psi)^* \Theta_\lambda \\
&= \frac{1}{\mathscr{L}^{n-1}} \bigwedge_{j=1}^{n} (-1)^{n-j} i_{d_n} \dots i_{d_{j+1}} i_{d_{j-1}} \dots i_{d_1} (\bar{\psi}^{-1} \circ \psi)^* \Theta_\lambda \\
&= \frac{1}{\mathscr{L}^{n-1}} \bigwedge_{j=1}^{n} (-1)^{n-j} i_{d_n} \dots i_{d_{j+1}} i_{d_{j-1}} \dots i_{d_1} \Theta_\lambda,
\end{aligned}$$

as required.

2. Analogously to the proof of Lemma 4, we find a chart expression of 1-form

$$i_{d_n} \dots i_{d_{j+1}} i_{d_{j-1}} \dots i_{d_1} \Theta_\lambda,$$

where Θ_λ is the principal Lepage equivalent (6). Using $dx^k \wedge \omega_j = \delta^k_j \omega_0$, we have

$$\begin{aligned}
\Theta_\lambda = &\left(\mathscr{L} - \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_j} - d_p \frac{\partial \mathscr{L}}{\partial y^\sigma_{pj}} \right) y^\sigma_j - \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} y^\sigma_{ij} \right) \omega_0 \\
&+ \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_j} - d_p \frac{\partial \mathscr{L}}{\partial y^\sigma_{pj}} \right) dy^\sigma \wedge \omega_j + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} dy^\sigma_i \wedge \omega_j.
\end{aligned}$$

Then,

$$
\begin{aligned}
& i_{d_{j-1}} \dots i_{d_1} \Theta_\lambda \\
& = \left(\mathscr{L} - \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_k} - d_p \frac{\partial \mathscr{L}}{\partial y^\sigma_{pk}} \right) y^\sigma_k - \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} y^\sigma_{ik} \right) dx^j \wedge \dots \wedge dx^n \\
& + \sum_{l=1}^{j-1} (-1)^{l-1} \left(\left(\frac{\partial \mathscr{L}}{\partial y^\sigma_k} - d_p \frac{\partial \mathscr{L}}{\partial y^\sigma_{pk}} \right) y^\sigma_l + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} y^\sigma_{il} \right) i_{d_{j-1}} \dots i_{d_{l+1}} i_{d_{l-1}} \dots i_{d_1} \omega_k \\
& + (-1)^{j-1} \left(\left(\frac{\partial \mathscr{L}}{\partial y^\sigma_k} - d_p \frac{\partial \mathscr{L}}{\partial y^\sigma_{pk}} \right) dy^\sigma + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} dy^\sigma_i \right) \wedge \left(i_{d_{j-1}} \dots i_{d_1} \omega_k \right) \\
& = \left(\mathscr{L} - \sum_{k=j}^{n} \left(\left(\frac{\partial \mathscr{L}}{\partial y^\sigma_k} - d_p \frac{\partial \mathscr{L}}{\partial y^\sigma_{pk}} \right) y^\sigma_k + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} y^\sigma_{ik} \right) \right) dx^j \wedge \dots \wedge dx^n \\
& + \sum_{l=1}^{j-1} (-1)^{l-1} \sum_{k=j}^{n} \left(\left(\frac{\partial \mathscr{L}}{\partial y^\sigma_k} - d_p \frac{\partial \mathscr{L}}{\partial y^\sigma_{pk}} \right) y^\sigma_l + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} y^\sigma_{il} \right) i_{d_{j-1}} \dots i_{d_{l+1}} i_{d_{l-1}} \dots i_{d_1} \omega_k \\
& + (-1)^{j-1} \sum_{k=j}^{n} \left(\left(\frac{\partial \mathscr{L}}{\partial y^\sigma_k} - d_p \frac{\partial \mathscr{L}}{\partial y^\sigma_{pk}} \right) dy^\sigma + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} dy^\sigma_i \right) \wedge \left(i_{d_{j-1}} \dots i_{d_1} \omega_k \right),
\end{aligned}
$$

and

$$
\begin{aligned}
& i_{d_{j+1}} i_{d_{j-1}} \dots i_{d_1} \Theta_\lambda \\
& = \left(-\mathscr{L} + \sum_{\substack{k=j \\ k \neq j+1}}^{n} \left(\left(\frac{\partial \mathscr{L}}{\partial y^\sigma_k} - d_p \frac{\partial \mathscr{L}}{\partial y^\sigma_{pk}} \right) y^\sigma_k + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} y^\sigma_{ik} \right) \right) dx^j \wedge \bigwedge_{l=j+2}^{n} dx^l \\
& + \sum_{l=1}^{j-1} (-1)^{l-1} \sum_{\substack{k=j \\ k \neq j+1}}^{n} \left(\left(\frac{\partial \mathscr{L}}{\partial y^\sigma_k} - d_p \frac{\partial \mathscr{L}}{\partial y^\sigma_{pk}} \right) y^\sigma_l + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} y^\sigma_{il} \right) \\
& \quad i_{d_{j+1}} i_{d_{j-1}} \dots i_{d_{l+1}} i_{d_{l-1}} \dots i_{d_1} \omega_k \\
& + (-1)^{j-1} \sum_{\substack{k=j \\ k \neq j+1}}^{n} \left(\left(\frac{\partial \mathscr{L}}{\partial y^\sigma_k} - d_p \frac{\partial \mathscr{L}}{\partial y^\sigma_{pk}} \right) y^\sigma_{j+1} + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} y^\sigma_{i,j+1} \right) i_{d_{j-1}} \dots i_{d_1} \omega_k \\
& + (-1)^{j} \sum_{k=j}^{n} \left(\left(\frac{\partial \mathscr{L}}{\partial y^\sigma_k} - d_p \frac{\partial \mathscr{L}}{\partial y^\sigma_{pk}} \right) dy^\sigma + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} dy^\sigma_i \right) \wedge \left(i_{d_{j+1}} i_{d_{j-1}} \dots i_{d_1} \omega_k \right).
\end{aligned}
$$

After the additional $n - j - 1$ steps, we obtain

$$
\begin{aligned}
& i_{d_n} \dots i_{d_{j+1}} i_{d_{j-1}} \dots i_{d_1} \Theta_\lambda \\
& = (-1)^{n-j} \left(\mathscr{L} dx^j + \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_j} - d_p \frac{\partial \mathscr{L}}{\partial y^\sigma_{pj}} \right) \omega^\sigma + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \omega^\sigma_i \right).
\end{aligned}
$$

3. From (10), it is evident that Λ_λ (9) is decomposable, $\pi^{3,1}$-horizontal, and obeys $h\Lambda_\lambda = \lambda$. It is sufficient to verify that Λ_λ is a Lepage form, that is $hi_\xi d\Lambda_\lambda = 0$ for an

arbitrary $\pi^{3,0}$-vertical vector field ξ on $W^3 \subset J^3Y$. This follows, however, by means of a straightforward computation using chart expression (10). Indeed, we have

$$
\begin{aligned}
d\Lambda_\lambda = {} & (1-n)\frac{1}{\mathscr{L}^n} d\mathscr{L} \wedge \bigwedge_{j=1}^{n} \left(\mathscr{L} dx^j + \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_j} - d_i \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \right) \omega^\sigma + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \omega^\sigma_i \right) \\
& + \frac{1}{\mathscr{L}^{n-1}} \sum_{k=1}^{n} (-1)^{k-1} d \left(\mathscr{L} dx^k + \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_k} - d_i \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} \right) \omega^\sigma + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} \omega^\sigma_i \right) \\
& \wedge \bigwedge_{j \neq k} \left(\mathscr{L} dx^j + \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_j} - d_i \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \right) \omega^\sigma + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \omega^\sigma_i \right),
\end{aligned}
$$

and the contraction of $d\Lambda_\lambda$ with respect to a $\pi^{3,0}$-vertical vector field ξ reads

$$
\begin{aligned}
i_\xi d\Lambda_\lambda = {} & (1-n)\frac{1}{\mathscr{L}^n} i_\xi d\mathscr{L} \bigwedge_{j=1}^{n} \left(\mathscr{L} dx^j + \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_j} - d_i \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \right) \omega^\sigma + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \omega^\sigma_i \right) \\
& - (1-n)\frac{1}{\mathscr{L}^n} d\mathscr{L} \wedge \sum_{l=1}^{n} (-1)^{l-1} \frac{\partial \mathscr{L}}{\partial y^\sigma_{jl}} \xi^\sigma_j \\
& \bigwedge_{j \neq l} \left(\mathscr{L} dx^j + \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_j} - d_i \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \right) \omega^\sigma + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \omega^\sigma_i \right) \\
& + \frac{1}{\mathscr{L}^{n-1}} \sum_{k=1}^{n} (-1)^{k-1} \left(i_\xi d\mathscr{L} dx^k + i_\xi d \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_k} - d_i \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} \right) \omega^\sigma \right. \\
& \left. - \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_k} - d_i \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} \right) \xi^\sigma_j dx^j + i_\xi d \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} \omega^\sigma_i - \xi^\sigma_i d \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} - \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} \xi^\sigma_{is} dx^s \right) \\
& \wedge \bigwedge_{j \neq k} \left(\mathscr{L} dx^j + \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_j} - d_i \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \right) \omega^\sigma + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \omega^\sigma_i \right) \\
& + \frac{1}{\mathscr{L}^{n-1}} \sum_{k=1}^{n} (-1)^{k-1} d \left(\mathscr{L} dx^k + \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_k} - d_i \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} \right) \omega^\sigma + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} \omega^\sigma_i \right) \\
& \wedge \sum_{l<k} (-1)^{l-1} \frac{\partial \mathscr{L}}{\partial y^\sigma_{il}} \xi^\sigma_i \bigwedge_{j \neq k,l} \left(\mathscr{L} dx^j + \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_j} - d_i \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \right) \omega^\sigma + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \omega^\sigma_i \right) \\
& + \frac{1}{\mathscr{L}^{n-1}} \sum_{k=1}^{n} (-1)^{k-1} d \left(\mathscr{L} dx^k + \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_k} - d_i \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} \right) \omega^\sigma + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ik}} \omega^\sigma_i \right) \\
& \wedge \sum_{l>k} (-1)^{l} \frac{\partial \mathscr{L}}{\partial y^\sigma_{il}} \xi^\sigma_i \bigwedge_{j \neq k,l} \left(\mathscr{L} dx^j + \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_j} - d_i \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \right) \omega^\sigma + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} \omega^\sigma_i \right).
\end{aligned}
$$

Hence, the horizontal part of $i_\xi d\Lambda_\lambda$ satisfies

$$\begin{aligned} hi_\xi d\Lambda_\lambda &= (1-n)i_\xi d\mathscr{L}\omega_0 - (1-n)\frac{1}{\mathscr{L}}\sum_{l=1}^{n}\frac{\partial\mathscr{L}}{\partial y^\sigma_{jl}}\xi^\sigma_j d_l\mathscr{L}\omega_0 + ni_\xi d\mathscr{L}\omega_0 \\ &\quad - \sum_{k=1}^{n}\left(\xi^\sigma_i d_k\frac{\partial\mathscr{L}}{\partial y^\sigma_{ik}} + \frac{\partial\mathscr{L}}{\partial y^\sigma_{ik}}\xi^\sigma_{ik}\right)\omega_0 - \sum_{k=1}^{n}\left(\frac{\partial\mathscr{L}}{\partial y^\sigma_k} - d_i\frac{\partial\mathscr{L}}{\partial y^\sigma_{ik}}\right)\xi^\sigma_k\omega_0 \\ &\quad + \frac{1}{\mathscr{L}}\sum_{k=1}^{n}(-1)^{k-1}d_s\mathscr{L}dx^s\wedge dx^k\wedge\sum_{l<k}(-1)^{l-1}\frac{\partial\mathscr{L}}{\partial y^\sigma_{il}}\xi^\sigma_i\bigwedge_{j\neq k,l}dx^j \\ &\quad + \frac{1}{\mathscr{L}}\sum_{k=1}^{n}(-1)^{k-1}d_s\mathscr{L}dx^s\wedge dx^k\wedge\sum_{l>k}(-1)^{l}\frac{\partial\mathscr{L}}{\partial y^\sigma_{il}}\xi^\sigma_i\bigwedge_{j\neq k,l}dx^j \\ &= -(1-n)\frac{1}{\mathscr{L}}\sum_{l=1}^{n}\frac{\partial\mathscr{L}}{\partial y^\sigma_{jl}}\xi^\sigma_j d_l\mathscr{L}\omega_0 - \frac{1}{\mathscr{L}}\sum_{k=1}^{n}\sum_{l\neq k}\frac{\partial\mathscr{L}}{\partial y^\sigma_{il}}\xi^\sigma_i d_s\mathscr{L}dx^s\wedge\omega_l \\ &= 0, \end{aligned}$$

where the identity $dx^k\wedge\omega_l = \delta^k_l\omega_0$ is applied.

□

The Lepage equivalent Λ_λ (9) is said to be the *Carathéodory form* associated to $\lambda\in\Omega^2_{n,X}W$.

4. The Carathéodory Form and Principal Lepage Equivalents in Higher-Order Theory

We point out that in the proof of Theorem 2, (a), the chart independence of Formula (9) is based on the principal Lepage equivalent Θ_λ (6) of a second-order Lagrangian, which is defined *globally*. Since for a Lagrangian of order $r\geq 3$, principal components of Lepage equivalents are, in general, *local* expressions (see the Introduction), we are allowed to apply the definition (9) for such class of Lagrangians of order r over a fibered manifold which assure the invariance of local expressions Θ_λ (1).

Consider now a *third-order* Lagrangian $\lambda\in\Omega^3_{n,X}W$. Then, the principal component Θ_λ of a Lepage equivalent of λ reads

$$\begin{aligned} \Theta_\lambda = \mathscr{L}\omega_0 &+ \left(\frac{\partial\mathscr{L}}{\partial y^\sigma_j} - d_p\frac{\partial\mathscr{L}}{\partial y^\sigma_{pj}} + d_pd_q\frac{\partial\mathscr{L}}{\partial y^\sigma_{pqj}}\right)\omega^\sigma\wedge\omega_j \\ &+ \left(\frac{\partial\mathscr{L}}{\partial y^\sigma_{kj}} - d_p\frac{\partial\mathscr{L}}{\partial y^\sigma_{kpj}}\right)\omega^\sigma_k\wedge\omega_j + \frac{\partial\mathscr{L}}{\partial y^\sigma_{klj}}\omega^\sigma_{kl}\wedge\omega_j. \end{aligned} \tag{12}$$

In the following lemma, we describe conditions for invariance of (12).

Lemma 5. *The following two conditions are equivalent:*

(a) Θ_λ *satisfies*

$$(\bar{\psi}^{-1}\circ\psi)^*\bar{\Theta}_\lambda = \Theta_\lambda.$$

(b) *For arbitrary two overlapping fibered charts on Y, (V,ψ), $\psi = (x^i,y^\sigma)$, and $(\bar{V},\bar{\psi})$, $\bar{\psi} = (\bar{x}^i,\bar{y}^\sigma)$,*

$$d_k\left(\left(\frac{\partial\mathscr{L}}{\partial y^\tau_{l_1l_2k}}\frac{\partial x^s}{\partial\bar{x}^p} - \frac{\partial\mathscr{L}}{\partial y^\tau_{l_1l_2s}}\frac{\partial x^k}{\partial\bar{x}^p}\right)\frac{\partial^2\bar{x}^p}{\partial x^{l_1}\partial x^{l_2}}\frac{\partial y^\tau}{\partial\bar{y}^\sigma}\right) = 0. \tag{13}$$

Proof. The equivalence of Conditions (a) and (b) follows from the chart transformation

$$\begin{aligned}
&\bar{x}^k = \bar{x}^k(x^s), \quad \bar{y}^\sigma = \bar{y}^\sigma(x^s, y^\nu),\\
&\bar{y}^\sigma_k = \frac{d}{d\bar{x}^k}(\bar{y}^\sigma) = \frac{\partial x^s}{\partial \bar{x}^k}\frac{\partial \bar{y}^\sigma}{\partial x^s} + \frac{\partial x^s}{\partial \bar{x}^k}\frac{\partial \bar{y}^\sigma}{\partial y^\nu}y^\nu_s,\\
&\bar{y}^\sigma_{ij} = \frac{d}{d\bar{x}^j}(\bar{y}^\sigma_i) = \frac{\partial x^t}{\partial \bar{x}^j}\frac{d}{dx^t}\left(\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial \bar{y}^\sigma}{\partial x^s} + \frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial \bar{y}^\sigma}{\partial y^\nu}y^\nu_s\right),\\
&\bar{y}^\sigma_{ijk} = \frac{d}{d\bar{x}^k}(\bar{y}^\sigma_{ij}) = \frac{\partial x^r}{\partial \bar{x}^k}\frac{d}{dx^r}\left(\frac{\partial x^t}{\partial \bar{x}^j}\frac{d}{dx^t}\left(\frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial \bar{y}^\sigma}{\partial x^s} + \frac{\partial x^s}{\partial \bar{x}^i}\frac{\partial \bar{y}^\sigma}{\partial y^\nu}y^\nu_s\right)\right),
\end{aligned}$$

applied to Θ_λ, where the Lagrange function $\mathscr{L}$ is transformed by (11) and the following identities are employed,

$$\begin{aligned}
&\bar{\omega}_j = \frac{\partial x^s}{\partial \bar{x}^j}\det\left(\frac{\partial \bar{x}^p}{\partial x^q}\right)\omega_s,\\
&\bar{\omega}^\sigma = \frac{\partial \bar{y}^\sigma}{\partial y^\nu}\omega^\nu, \quad \bar{\omega}^\sigma_i = \frac{\partial \bar{y}^\sigma_i}{\partial y^\nu}\omega^\nu + \frac{\partial \bar{y}^\sigma_i}{\partial y^\nu_p}\omega^\nu_p, \quad \bar{\omega}^\sigma_{ij} = \frac{\partial \bar{y}^\sigma_{ij}}{\partial y^\nu}\omega^\nu + \frac{\partial \bar{y}^\sigma_{ij}}{\partial y^\nu_p}\omega^\nu_p + \frac{\partial \bar{y}^\sigma_{ij}}{\partial y^\nu_{pq}}\omega^\nu_{pq}.
\end{aligned}$$

□

Theorem 3. *Suppose that a fibered manifold $\pi : Y \to X$ and a non-vanishing third-order Lagrangian $\lambda \in \Omega^3_{n,X}W$ satisfy condition* (13)*. Then, Λ_λ* (9)*, where Θ_λ is given by* (12) *and defines a differential n-form on $W^5 \subset J^5Y$, which is a Lepage equivalent of λ, decomposable and $\pi^{5,2}$-horizontal. In a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, on W, Λ_λ has an expression*

$$\begin{aligned}
\Lambda_\lambda = \frac{1}{\mathscr{L}^{n-1}}\bigwedge_{j=1}^{n}\Bigg(&\mathscr{L}dx^j + \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_j} - d_p\frac{\partial \mathscr{L}}{\partial y^\sigma_{pj}} + d_pd_q\frac{\partial \mathscr{L}}{\partial y^\sigma_{pqj}}\right)\omega^\sigma\\
&+ \left(\frac{\partial \mathscr{L}}{\partial y^\sigma_{ij}} - d_p\frac{\partial \mathscr{L}}{\partial y^\sigma_{ipj}}\right)\omega^\sigma_i + \frac{\partial \mathscr{L}}{\partial y^\sigma_{ikj}}\omega^\sigma_{ik}\Bigg)
\end{aligned} \tag{14}$$

Proof. This is an immediate consequence of Lemma 5 and the procedure given by Lemma 4 and Theorem 2. □

Note that, according to Lemma 5, characterizing obstructions for the principal component Θ_λ (12) to be a global form, the Carathéodory form (14) is well-defined for third-order Lagrangians on fibered manifolds, which satisfy condition (13). Trivial cases when (13) holds identically include: (i) Lagrangians *independent* of variables y^σ_{ijk}; and (ii) fibered manifolds with bases endowed by smooth structure with *linear* chart transformations. An example of (ii) are fibered manifolds over the two-dimensional open *Möbius strip* (for details, see [17]).

Suppose that a pair (λ, π), where $\pi : Y \to X$ is a fibered manifold and $\lambda \in \Omega^r_{n,X}W$ is a Lagrangian on $W^r \subset J^rY$, induces the *invariant* principal component Θ_λ (1) of a Lepage equivalent of λ, with respect to fibered chart transformations on W. We call n-form Λ_λ on W^{2r-1},

$$\Lambda_\lambda = \frac{1}{\mathscr{L}^{n-1}}\bigwedge_{j=1}^{n}(-1)^{n-j}i_{d_n}\dots i_{d_{j+1}}i_{d_{j-1}}\dots i_{d_1}\Theta_\lambda, \tag{15}$$

where Θ_λ is given by (1), the *Carathéodory form* associated to Lagrangian $\lambda \in \Omega^r_{n,X}W$.

In a fibered chart (V, ψ), $\psi = (x^i, y^\sigma)$, on W, Λ_λ has an expression

$$\Lambda_\lambda = \frac{1}{\mathscr{L}^{n-1}} \bigwedge_{j=1}^{n} \left(\mathscr{L} dx^j + \sum_{s=0}^{r-1} \left(\sum_{k=0}^{r-1-s} (-1)^k d_{p_1} \dots d_{p_k} \frac{\partial \mathscr{L}}{\partial y^\sigma_{i_1 \dots i_s p_1 \dots p_k j}} \right) \omega^\sigma_{i_1 \dots i_s} \right). \quad (16)$$

5. Example: The Carathéodory Equivalent of the Hilbert Lagrangian

Consider the fibered manifold Met$X$ of metric fields over an n-dimensional manifold X (see [18] for geometry of Met$X$). In a chart (U, φ), $\varphi = (x^i)$, on X, section $g : X \supset U \to$ Met$X$ is expressed by $g = g_{ij} dx^i \otimes dx^j$, where g_{ij} is symmetric and regular at every point $x \in U$. An induced fibered chart on the second jet prolongation J^2Met$X$ reads (V, ψ), $\psi = (x^i, g_{jk}, g_{jk,l}, g_{jk,lm})$.

The *Hilbert Lagrangian* is an odd-base n-form defined on J^2Met$X$ by

$$\lambda = \mathscr{R} \omega_0, \quad (17)$$

where $\mathscr{R} = R\sqrt{|\det(g_{ij})|}$, $R = R(g_{ij}, g_{ij,k}, g_{ij,kl})$ is the *scalar curvature* on J^2Met$X$, and $\mu = \sqrt{|\det(g_{ij})|}\omega_0$ is the *Riemann volume element*.

The principal Lepage equivalent of λ (17) (cf. Formula (6)) reads

$$\Theta_\lambda = \mathscr{R} \omega_0 + \left(\left(\frac{\partial \mathscr{R}}{\partial y^\sigma_j} - d_p \frac{\partial \mathscr{R}}{\partial y^\sigma_{pj}} \right) \omega^\sigma + \frac{\partial \mathscr{R}}{\partial y^\sigma_{ij}} \omega^\sigma_i \right) \wedge \omega_j, \quad (18)$$

and it is a globally defined n-form on J^1Met$X$. (18) was used for a study of the Einstein equations as a system of *first-order* partial differential equations (see [23]). Another Lepage equivalent for a second-order Lagrangian in field theory which could be studied in this context is given by (10),

$$\Lambda_\lambda = \frac{1}{\mathscr{R}^{n-1}} \bigwedge_{k=1}^{n} \left(\mathscr{R} dx^k + \left(\frac{\partial \mathscr{R}}{\partial g_{ij,k}} - d_l \frac{\partial \mathscr{R}}{\partial g_{ij,kl}} \right) \omega_{ij} + \frac{\partial \mathscr{R}}{\partial g_{ij,kl}} \omega_{ij,l} \right),$$

where $\omega_{ij} = dg_{ij} - g_{ij,s} dx^s$, $\omega_{ij,l} = dg_{ij,l} - g_{ij,ls} dx^s$. Using a chart expression of the scalar curvature, we obtain

$$\begin{aligned} \Lambda_\lambda = \frac{1}{\mathscr{R}^{n-1}} \bigwedge_{k=1}^{n} \Big(& \mathscr{R} dx^k + \frac{1}{2}\sqrt{g} \left(g^{qp} g^{si} g^{jk} - 2 g^{sq} g^{pi} g^{jk} + g^{pi} g^{qj} g^{sk} \right) g_{pq,s} \omega_{ij} \\ & + \sqrt{g} \left(g^{il} g^{kj} - g^{kl} g^{ji} \right) \omega_{ij,l} \Big). \end{aligned}$$

This is the Carathéodory equivalent of the Hilbert Lagrangian.

6. Discussion

The Lepage equivalents of Lagrangians belong to the basic themes in the global variational theory (in both fibered spaces and the theory for submanifolds) and its applications in differential geometry and mathematical physics. These objects completely describe a global variational problem by means of intrinsic geometric operations. Since the pioneering works by García [6] and Krupka [7,11,12], there have been several attempts at finding new Lepage equivalents in higher-order field theory (cf. [3,20,21,24]).

In this work, we contribute to this area namely by establishing a new relationship between the Poincaré–Cartan and Carathéodory forms and, on this basis, by generalizing the Carathéodory form for higher-order Lagrangians in field theory. For the order $r = 2$, the result is general, whereas, for $r \geq 3$, the Carathéodory form in higher-order theory is obtained under specific assumptions on global properties, defined by the initial Lagrangian and the underlying topological space under consideration. In the forthcoming works, these results will be further studied in the sense of the mentioned topological obstructions and applied in physical field theories.

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